Einstein's Special Theory of Relativity Part-II

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"Life is like riding a bicycle. To keep your balance, you must keep moving."
— Albert Einstein

In the earlier article on special theory of relativity, we have studied postulates of the special relativity theory, how to obtain Lorentz transformation equations, deduction of length contraction and time dilation formulae. Below I list all the points which are covered in the present article.

I. OUTLINE

- Section II: Summary of last note on relativity theory
- Section III: Deduction of time dilation formula using a thought experiment
- Section IV: Relativistic velocity addition law

II. SUMMARY OF LAST NOTE ON RELATIVITY

- **Lorentz transformation equations**:

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$$x' = \gamma(x - ut), \tag{1}$$

$$y' = y, \tag{2}$$

$$z' = z, \tag{3}$$

$$t' = \gamma(t - ux/c^2), \tag{4}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ with $\beta = u/c$.

- **Inverse Lorentz transformation equations**: Replace $u \to -u$.

$$x = \gamma(x' + ut), \tag{5}$$

$$y = y', \tag{6}$$

$$z = z', \tag{7}$$

$$t = \gamma(t' + ux'/c^2). \tag{8}$$

- **Length contraction and time dilation**:

  ⇒ Length contraction is given by

$$\boxed{l = l_0/\gamma = l_0\sqrt{1-\beta^2}}\,, \tag{9}$$

  where l_0 is rest length (proper length) of a rod and l is measured length by an observer for whom rod is in relative motion with speed u.

  ⇒ Time dilation is given by

$$\boxed{\tau = \gamma\tau_0 = \frac{\tau_0}{\sqrt{1-\beta^2}}}\,, \tag{10}$$

  where τ_0 is time interval measured in the rest frame of the clock (proper time interval) and τ is measured time by an stationary frame observer for whom clock is moving with speed u with respect to (w.r.t.) him. *Thus, clock in motion relative to an observer runs slow than clock at rest relative to the observer.*

- **Proper frame:** We call a frame as proper frame in which observed body is at rest and length and time interval measured in proper frame is called respectively as *proper length* and *proper time interval.*

III. DERIVATION OF TIME DILATION USING A THOUGHT EXPERIMENT

Here we rederive time dilation result using a thought experiment without using Lorentz transformation equations. We mainly consider the second postulate of the relativity theory.

In Fig. (1), we have considered a S' frame a moving train and S frame ground. Train is moving with speed u with respect to ground. Moving frame and ground observers sees different locations of the beginning and end of the event, i.e., turning on light from the source and reflect back from the mirror to point R. The S frame observer sees that light has traveled greater path to reach at point R hence he measures more time taken to travel light from source to end point R.

Let time interval, from start to end of the event, measured by the moving train observer is τ_0 and for ground frame observer τ. We obtain

$$\tau_0 = 2QS/c \quad \text{and} \quad \tau = (PQ + QR)/c, \tag{11}$$

where c denotes the speed of light. Now using Pythagorean theorem in Fig 1(b) we can write,

$$(QS)^2 = (PQ)^2 - (PS)^2 = (QR)^2 - (SR)^2,$$

in above relation note that distance PS is traveled with speed u while hypotenuse distance PQ(or QR) is traveled with light speed c in same time. Now Using (11) in (12), we obtain

$$\begin{aligned}
\tau_0/\tau &= 2QS/(PQ + QR) = 2QS/2PQ \\
&= \sqrt{(PQ)^2 - (PS)^2}/PQ \\
&= \sqrt{1 - (PS/PQ)^2} \\
&= \sqrt{1 - (u/c)^2} \\
&= \sqrt{1 - \beta^2}
\end{aligned}$$

Or

$$\tau = \frac{\tau_0}{\sqrt{1 - \beta^2}}, \tag{12}$$

which is same as obtained using Lorentz transformation rules.

Home Work: Obtain length contraction formula without using Lorentz transformation equations. You need to think a thought experiment and use the above time dilation result. Length contraction is a consequence of time dilation which we have obtained above.

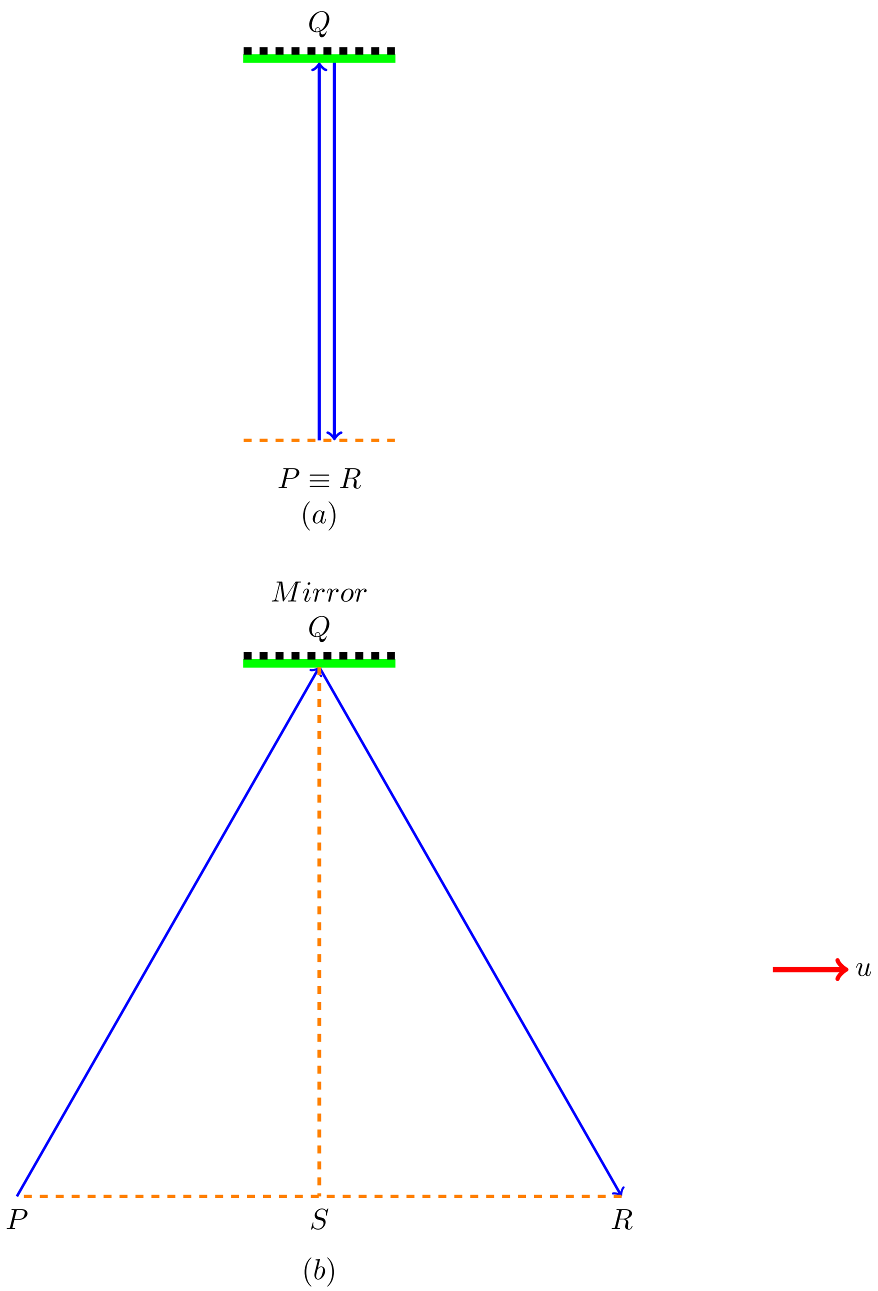

FIG. 1: Figure demonstrates a thought experiment (turning on a flashlight in a moving train or bus). Q represents the mirror (fixed on the ceiling) which reflects back the light. (a) Observer in a moving train (S' frame) sees the path of the light from P to Q and back from Q to $P \equiv R$. For this observer start and end location of the event are same, i.e., $P \equiv R$. (b)Denotes light path for ground observer. Train moving to right with velocity u with respect to ground. For ground observer start and end point of event are not same.

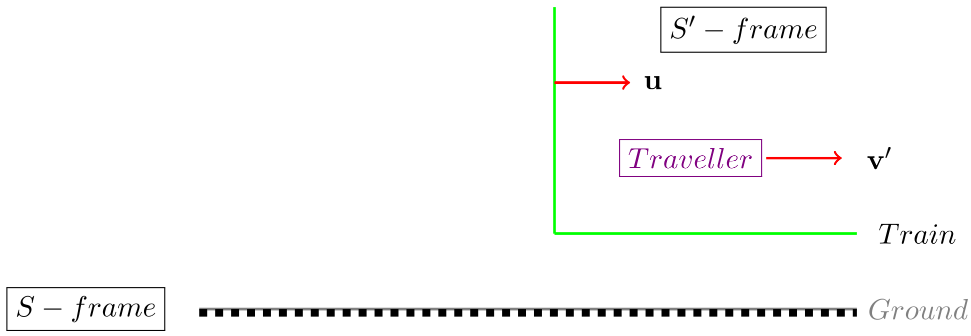

FIG. 2: A schematic diagram. S' frame (Train) moving with constant velocity $\mathbf{u}$ relative to S frame (Ground) in the +ive $x-$direction, where $\mathbf{u} = (u, 0, 0)$. Passenger is moving with velocity $\mathbf{v}'$ with respect to train .

IV. RELATIVISTIC VELOCITY ADDITION LAW

In order to obtain relativistic velocity addition, we consider the following points (see Fig. (2)):

- S-frame: Ground
- S'-frame: Train moving with velocity $\mathbf{u}$ with respect to ground.
- Passenger moving with velocity $\mathbf{v}'$ with respect to ground, where $\mathbf{v}' \equiv (v'_x, v'_y, v'_z)$.
- Let velocity of passenger observed by ground frame observer is $\mathbf{v}$, where $\mathbf{v} \equiv (v_x, v_y, v_z)$.
- After some time passenger's location for moving train observer is given by $x' = v'_x t'$ and passenger's location for ground observer is given by $x = v_x t$.

Writing Lorentz transformation rules

$$x' = \gamma(x - ut) \quad \text{and} \quad t' = \gamma(t - ux/c^2) \tag{13}$$

Using $x' = v'_x t'$, we obtain

$$\gamma(x - ut) = v'_x t' = v'_x \gamma(t - ux/c^2). \tag{14}$$

Or

$$(x - ut) = v'_x(t - ux/c^2), \tag{15}$$

which gives

$$x = \frac{v'_x + u}{1 + v'_x u/c^2} t \ .$$

Again using $x = v_x t$, we obtain

$$\boxed{v_x = \frac{v'_x + u}{1 + v'_x u/c^2}}. \tag{16}$$

For the y component of the velocity, we consider that a body is moving parallel to the y'-direction in the S' frame. It location at times t'_1 and t'_2 are given by y'_1 and y'_2, and hence y-component of velocity is given by

$$v'_y = \frac{y'_2 - y'_1}{t'_2 - t'_2} = \frac{\Delta y'}{\Delta t'}.$$

Using Lorentz transformation we obtain that,

$$\begin{aligned} t'_2 - t'_1 &= \gamma[(t_2 - t_1) - (x_2 - x_1)u/c^2] = \gamma[\Delta t - \Delta x u/c^2], \\ y'_2 - y'_1 &= y_2 - y_1. \end{aligned} \tag{17}$$

Now, using above relations, we obtain

$$\begin{aligned} \frac{y'_2 - y'_1}{t'_2 - t'_1} &= \frac{\Delta y\sqrt{1-\beta^2}}{\Delta t - \Delta x u/c^2} \\ &= \frac{(\Delta y/\Delta t)\sqrt{1-\beta^2}}{1 - (\Delta x/\Delta t)u/c^2}. \end{aligned}$$

Or

$$\boxed{v'_y = \frac{v_y\sqrt{1-\beta^2}}{1 - v_x u/c^2}}. \tag{18}$$

For the S-frame observer v_y is obtained by replacing $u \to -u$ and interchanging primed quantities to unprimed quantities and vice-versa, we obtain

$$\boxed{v_y = \frac{v'_y\sqrt{1-\beta^2}}{1 + v'_x u/c^2}}. \tag{19}$$

Using similar argument, we obtain the z-component of velocity as

$$\boxed{v_z = \frac{v'_z\sqrt{1-\beta^2}}{1 + v'_x u/c^2}}. \tag{20}$$

We summarize below the velocity transformation rule

- **Primed frame observer (a train moving w.r.t. ground in our example)**:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}, \tag{21}$$
$$v'_y = \frac{v_y\sqrt{1-\beta^2}}{1 - v_x u/c^2}, \tag{22}$$
$$v'_z = \frac{v_z\sqrt{1-\beta^2}}{1 - v_x u/c^2}, \tag{23}$$

- **Unprimed (stationary) frame observer (ground in our example)**

$$v_x = \frac{v'_x + u}{1 + v'_x u/c^2}, \tag{24}$$
$$v_y = \frac{v'_y\sqrt{1-\beta^2}}{1 + v'_x u/c^2}, \tag{25}$$
$$v_z = \frac{v'_z\sqrt{1-\beta^2}}{1 + v'_x u/c^2}, \tag{26}$$

Home Work: Take limit $u/c \ll 1$ and see what happens to above velocity transformation rules.

Note: *Taking time derivative of velocity components in above transformations, one can obtain acceleration transformation rule for the two frames. One can verify that acceleration components are not same in both the frame. It was same in Newtonian relativity. Thus for the special relativity case, though the frame is unaccelerated but the moving object may have some acceleration w.r.t. the frames.*

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[1] R. Resnick, *Introduction to Special Relativity*, Wiley-VCH, (1968) .

[2] A. Beiser, *Concepts of modern physics*, Tata McGraw-Hill Education, (2003).